Research Statement

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1 Introduction

My main research area is in group theory and representation theory of finite groups. The study of group theory was motivated by geometry and the desire to understand symmetry, as well as by number theory and the theory of algebraic equations. Symmetry occurs in art, communication networking, crystals and atoms, and the body structure of most organisms. Consequently, group theory is a highly active field of research, with applications in pure and applied mathematics, chemistry, physics, cryptography, and coding theory. Representation theory gives a link between arbitrary groups in nature and a collection of well understood objects, namely matrices. These representations decompose into irreducible representations. We can then take the trace of the image of group elements under our irreducible representations, and this affords an irreducible character of our group. We are interested in extracting as much information from the collection of irreducible characters as possible, in an effort to identify the structure of arbitrary groups.

My thesis research involves the fields of values irreducible characters of finite groups. The classification of finite simple groups tells us that every finite simple group is either cyclic of prime order, a Sporadic group, an alternating group, or a group of Lie type. I focus on the last such class of simple groups, which can be divided further into two types, namely the classical groups of Lie type and the exceptional groups of Lie type. The classical groups can be realized nicely as groups of matrices over finite fields, and thus, we can take a hands on approach when constructing the irreducible characters. However, the exceptional groups do not have such nice realizations, and we must take a more theoretical approach.

To be more precise, a *complex representation* of a finite group G is a homomorphism $\Phi : G \to \mathrm{GL}_n(\mathbb{C})$, where $\mathrm{GL}_n(\mathbb{C})$ is the group of invertible n by n matrices with entries in $\mathbb{C}$, the complex numbers. Such a representation uniquely defines a $\mathbb{C}G$ module V, with the action of G on V given by the representation. The representation Φ is then said to be irreducible if V is irreducible as a $\mathbb{C}G$ module, i.e. V has exactly two $\mathbb{C}G$ submodules. Furthermore, every $\mathbb{C}G$ module W has a composition series $0 = W_1 \leq W_2 \leq \cdots \leq W_m = W$ with W_{i+1}/W_i irreducible. In this sense, we may decompose every $\mathbb{C}G$ representation into a direct sum of irreducible representations.

Now, we may consider the class function $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \mathrm{tr}(\Phi(g))$. We call this the character afforded by the representation Φ, and denote the collection of all irreducible characters of G by $\mathrm{Irr}(G)$. It is well known (see for example [12]) that χ takes values in $\mathbb{Q}(\zeta)$, where ζ is a primitive $|G|$-th root of unity, and $|G|$ is the order of the group G. However, if we examine the character tables of various groups, it is clear that the fields of values of the irreducible characters is generally much smaller. For example, every irreducible character of a symmetric group, S_n, takes values solely in the rational numbers, despite having group order $|S_n| = n!$. We define a finite group G to be *m-rational* if $[\mathbb{Q}(\chi) : \mathbb{Q}] \mid m$ for all irreducible characters $\chi \in \mathrm{Irr}(G)$. In section 2, I will discuss the motivation for this definition, as well as my results on the structure of m-rational groups.

Modular representation theory was developed in the 1940's by Richard Brauer. Modular representation theory has applications in various areas of mathematics, including group theory, number theory, and algebraic geometry. It is critically used in the proof of the classification of finite simple groups, in Wiles' proof of Fermat's last theorem, and in Quillen's proof of the Adams conjecture. In this branch of representation theory, we now consider the representations $\Phi : G \to \mathrm{GL}(V)$, where V is a finite-dimensional vector space over a field $\mathbb{F}$ of characteristic $\ell > 0$. As before, we collect the *ℓ-Brauer characters* afforded by the irreducible modular representations into a finite set, $\mathrm{IBr}_\ell(G)$. We define a finite group G to be *ℓ-Brauer m-rational* if $[\mathbb{Q}(\phi) : \mathbb{Q}] \mid m$ for all $\phi \in \mathrm{IBr}_\ell(G)$. In section 3, we explain that m-rationality and ℓ-Brauer m-rationality are independent, in that neither implies the other, and we give our results on ℓ-Brauer m-rational groups.

2 m-Rational Groups

The study of rational groups dates back to the 1970's when R. Gow determined the possible cyclic composition factors of any solvable rational group, [10]. Just over a decade later, W. Feit and G. Seitz found that the non-abelian composition factor of any (non-solvable) rational group is either an alternating group or one of five simple groups of Lie type, [5]. Using this result of Feit and Seitz, J. Thompson found an upper bound on the order of any cyclic composition of factor of a rational group, [18]. Recently, J. McKay posed the following question:

Problem 1 *Is it possible to determine the potential composition factors of quadratic rational groups?*

In 2013, J. Tent solved this problem in that case that G is a solvable group, [17]. He determined that the possible composition factors of a solvable quadratic rational group are cyclic groups, C_p with $p \leq 13$, and $p \neq 11$. In my thesis, I handle the case when G is not solvable, and give a list of the possible non-abelian composition factors of a non-solvable quadratic rational group.

Theorem 1 *Suppose that G is a non-solvable quadratic rational group. Let S be a non-abelian composition of G. Then S is one of the following:*

- *An alternating group A_n, $n \geq 5$.*
- *One of 20 Sporadic groups.*
- *A group of Lie type from the set:*

$$\{L_2(7), L_2(11), L_2(16), L_2(27), L_3(3), L_3(4), L_4(3),$$

$$U_3(4), U_3(5), U_3(8), U_4(2), U_4(3), U_5(2), U_5(4), U_6(2),$$

$$S_4(4), S_6(2), S_6(3), S_8(2), O_7(3), O_8^+(2), O_8^+(3), O_8^-(2), O_{10}^-(2),$$

$${}^2E_6(2), F_4(2), {}^2F_4(2)', G_2(2)', G_2(3), G_2(4), {}^2G_2(3)', {}^3D_4(2)\}.$$

Moreover, each of these groups S does occur as a composition factor of some quadratic rational group.

To prove this statement, we first observed that S_n, the symmetric group on n letters, is rational, and thus A_n for $n \geq 5$ occurs as a composition factor of some quadratic rational group. Thus, our problem was reduced to the case that S is a finite group of Lie type or a Sporadic group. We wished to translate our assumption about G, namely that every irreducible character, $\chi \in \mathrm{Irr}(G)$, satisfies $[\mathbb{Q}(\chi) : \mathbb{Q}] \mid 2$, into a statement about the irreducible characters of S. Using nice applications of Clifford theory and Galois theory, we are able to formulate this reduction to simple groups as the following lemma:

Lemma 1 *Assume the hypotheses of Theorem 1. Then $[\mathbb{Q}(\alpha) : \mathbb{Q}] \mid 2|\mathrm{Out}(S)|$ for all $\alpha \in \mathrm{Irr}(S)$, where $\mathrm{Out}(S)$ is the group of outer automorphisms of S.*

Using GAP, we could easily verify which Sporadic groups satisfied the conclusions of our reduction lemma. Moreover, those that satisfied the conclusion, did occur as a composition factor of some quadratic rational group. Using Deligne-Lusztig theory, I then considered semisimple irreducible characters of S which take on highly irrational values. For example, if $S = \mathrm{PSL}_n(q) = \mathrm{PSL}_n(p^f)$, is a composition factor of a quadratic rational group, we can construct an irreducible character χ, which satisfies

$$\frac{\phi\left(\frac{q^n-1}{(n,q-1)(q-1)}\right)}{n} \Bigg| [\mathbb{Q}(\chi):\mathbb{Q}] \Bigg| 2|\mathrm{Out}(S)| = \begin{cases} 2f(n,q-1) & n=2 \\ 4f(n,q-1) & n>2 \end{cases}.$$

Using such divisibility relations, we are able to bound the rank of S, and then bound the field size, q. This gives a finite list of groups $S = \mathrm{PSL}_n(q)$ that can occur as a composition factor of a quadratic rational group. Applying this general strategy to the remaining groups of Lie type, we can finish the proof of Theorem 1.

An astounding feature of our reduction Lemma 1 is that the argument extends to m-rational groups. That is, we have the following reduction lemma:

Lemma 2 *Suppose that S is a non-abelian composition factor of an m-rational group. Then $[\mathbb{Q}(\alpha) : \mathbb{Q}] \mid m|\mathrm{Out}(S)|$ for all $\alpha \in \mathrm{Irr}(S)$.*

Combining this with the divisibility criteria we obtained from examining semisimple characters, we can state the following theorem:

Theorem 2 *Let S be a non-abelian composition factor of an m-rational group G. Then S is either an alternating group, a Sporadic group, or belongs to some finite subset of the finite groups of Lie type, call it $\mathcal{F}(m)$.*

2.1 Related Future Work

Though the cyclic composition factors of solvable quadratic rational groups have been classified, it is still an open problem to determine the possible cyclic composition factors of non-solvable quadratic rational groups. This problem appears to be extremely difficult, as the result could not be fully completed in the rational case. However, in the future, I may closely examine the work of J. Thompson in [18] in the hopes of determining an upper bound on the order of the cyclic composition factors of non-solvable quadratic rational groups.

Another direction that my thesis results may point is toward the Guralnick-Thompson conjecture, now a theorem proved by D. Frohardt and K Magaard, [6]. This conjecture states that the non-abelian composition factors of the monodromy group of coverings of a genus m compact Riemann surface is either an alternating group or a member of a finite set $\mathcal{E}(m)$. Though this statement is eerily similar to that in my thesis, it is not yet known if there is a connection between the results. In the future, I hope to find a bridge between this seemingly geometric theorem and my purely group and character theoretic proposition.

3 ℓ-Brauer m-Rational Groups

More recently, I have begun to tackle the Brauer analogue of the m-rational problem:

Problem 2 *Determine the potential non-abelian compositions factors of ℓ-Brauer m-rational groups.*

The complex and Brauer problems are independent of one another, in that neither implies the other. For example, $\mathrm{SL}_2(7)$ is a quadratic rational group, but is not 7-Brauer quadratic rational. Likewise, A_5 is 5-Brauer rational, but is not rational. Nevertheless, our preliminary results and conjectures are similar in flavor to those found in the complex case, however, the proofs require substantially more work. For instance, one critical fact that we used in proving the main reduction lemma from section two is that the Galois conjugate of a complex irreducible character is again an irreducible complex character. This is *not* the case for modular characters. To see this, we may again consider $\mathrm{SL}_2(7)$ with $\ell = 7$. Thus, to maneuver past this drawback, we demand an assumption on the Brauer characters of our composition factors, namely liftability. An irreducible Brauer character $\alpha \in \mathrm{IBr}_\ell(S)$ is *liftable* to a complex character if there exists an irreducible complex character $\chi \in \mathrm{Irr}(S)$, such that its restriction to the ℓ-regular classes, χ°, is equal to α. With this assumption included, we obtain the following reduction lemma:

Lemma 3 *Suppose S is a non-abelian composition factor of an ℓ-Brauer m-rational group, and let $\alpha \in \mathrm{IBr}_\ell(S)$. If α is liftable, then $[\mathbb{Q}(\alpha) : \mathbb{Q}] \mid m|\mathrm{Out}(S)|$.*

This liftability constraint is necessary. A_8, the alternating group on eight letters, is a composition factor of the 2-Brauer rational group S_8, but has an irreducible character $\alpha \in \mathrm{IBr}_2(A_8)$ that does not lift to a complex character, and $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$.

At this point, one might wonder if such non-trivial liftable characters exist. It turns out that specific semisimple complex characters of groups of Lie type in characteristic $p \neq \ell$ are irreducible Brauer characters when restricted to the ℓ-regular classes. Returning to our $S = \mathrm{PSL}_n(q)$ example, we are able to conclude that if S is a non-abelian composition factor of an ℓ-Brauer m-rational group with $\ell \nmid q$, then

$$\left.\frac{\phi\left(\frac{q^n-1}{(n,q-1)(q-1)}\right)}{n}\right| m|\,\mathrm{Out}(S)| \qquad \text{or} \qquad \left.\frac{\phi\left(\frac{q^{n-1}-1}{(n,q-1)}\right)}{n-1}\right| m|\,\mathrm{Out}(S)|.$$

Using such divisibility relations, we expect to obtain the following result:

Conjecture 1 *Suppose that S is a non-abelian composition factor of an ℓ-Brauer m-rational group. Then S is one of the following:*

- *An alternating group, A_n, $n \geq 5$*
- *A Sporadic group*
- *A group of Lie type in characteristic ℓ*
- *A group of Lie type in characteristic $p \neq \ell$ from a finite set $\mathcal{F}(m)$.*

3.1 Related Future Work

My short-term goals are, of course, to prove the conjecture. Furthermore, I wish to classify, explicitly, the groups of Lie type, S, in characteristic $p \neq \ell$ that occur as a composition factor of some ℓ-Brauer rational group (i.e. $m = 1$). Additionally, I want to determine all primes ℓ, for which this occurs. For example, $L_2(4)$ occurs as a composition of an ℓ-Brauer rational group for all primes ℓ, namely S_5, whereas $L_2(25)$ can only occur as a composition factor of an ℓ-Brauer rational group for $\ell = 13$. A long-term goal is to determine the groups of Lie type in characteristic ℓ that occur as a composition factor of some ℓ-rational group. This classification will require significantly more work, as irreducible complex characters generally no longer remain irreducible Brauer characters when restricted to the ℓ-regular classes. Thus, I will need to deeply investigate the irreducible F-represenations of groups of Lie type with defining characteristic equal to that of $\mathrm{Char}(F)$.

4 Other Future Directions

One of the main goals of representation theory is to find correlations between representation theoretic properties of a group and its structure. For instance, Navarro and Tiep prove that a finite group G has exactly two irreducible rational-valued characters if and only if G has exactly two rational conjugacy classes, [15]. Further, it is known that if G is a finite group whose splitting field is a cyclic extension of $\mathbb{Q}$, then the number of rational characters is equal to the number of rational conjugacy classes in G. I hope to translate my research concerning the fields of values of irreducible characters into statements characterizing the structure of the conjugacy classes of finite groups.

More recently, similar statements have been proven regarding Brauer characters. For example, Navarro and Tiep show that a finite group G has no non-trivial ℓ-Brauer rational characters if and only if G has no non-trivial rational ℓ'-elements, [14]. As in the ordinary case, I hope to extend these results to statements comparing the number of irreducible ℓ-Brauer m-rational characters to the number of m-rational ℓ-regular conjugacy classes.

5 Undergraduate Research

As an undergraduate, I was involved in an REU at the State University of New York at Potsdam. Therefore, I know first-hand that undergraduate research can inspire young students to pursue higher degrees and careers in mathematics. Though, I did not research algebra specifically at my REU, I understand that algebra, and more specifically, group theory is an exceptional area to research as an undergraduate. Finite group theory and linear algebra tend to be very hands-on, and we can integrate technology such as GAP or CHEVIE to study the structure of groups. Further, this research with computer systems opens the door to various career opportunities in industry or other STEM fields.

I have an active history of promoting research in mathematics, and particularly in algebra. I encourage my calculus students to enroll in upper division courses such as abtract algebra and linear algebra. At recruitment workshops, I talk about the exciting applications of group theory to other areas of mathematics, and motivate new researchers to take a closer look at representation theory. In the future, I will continue to push young mathematicians toward representation theory and its beautiful applications, as well as non-mathematicians toward the study of algebra and mathematics in general.

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